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A Lower Bound on the Norm of Interpolation with an Extended Tchebycheff System

THEODORE KILGORE

Department of Mathematics, Auburn University, Alabama 36849-3501, U.S.A.

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If $\mathbf{Y}$ is an $n+1$-dimensional subspace of $C[a, b]$, spanned by an extended Tchebycheff system, it is possible to obtain an interpolating projection P onto $\mathbf{Y}$, using any set of nodes

$$t_0, ..., t_n$$

satisfying

$$a = t_0 < \cdots < t_n = b,$$

by defining *fundamental functions*

$$y_0, ..., y_n$$

in $\mathbf{Y}$ such that

$$y_i(t_j) = \delta_{ij}(\text{Kronecker delta}), \qquad \text{for} \quad i, j \in \{0, ..., n\},$$

and setting

$$Pf = \sum_{i=0}^{n} f(t_i)\, y_i, \qquad \text{for} \quad f \in C[a, b].$$

It is easily seen that

$$\|P\| = \left\| \sum_{i=0}^{n} |y_i| \right\|.$$

Denoting by X_i the function in $\mathbf{Y}$ whose restriction to the interval $[t_{i-1}, t_i]$, for $i \in \{1, ..., n\}$, agrees with

$$\sum_{j=0}^{n} |y_j|,$$

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and by T_i the maximum of X_i in $[t_{i-1}, t_i]$, we observe that

$$\|P\| = \max\{X_1(T_1),..., X_n(T_n)\}.$$

An old conjecture of Bernstein [1] and Erdös [3] states that, if $\mathbf{Y}$ is the space of polynomials of degree n or less, $\|P\|$ is minimized when the local maximum values

$$X_1(T_1),..., X_n(T_n)$$

are equal, which condition occurs at a unique placement of the nodes. If moreover the nodes are not thus strategically placed, at least one of these values exceeds the minimal norm, which in turn exceeds another of them.

The proof of these conjectures follows from the fact that the following two conditions are satisfied for arbitrary choices of nodes:

$$\det(\partial X_i(T_i)/\partial t_j)_{\substack{i=1\\ i\neq k}}^{m}\,{}_{j=1}^{m-1} \neq 0, \tag{1}$$

for $m \in \{2,..., n\}$ and $k \in \{1,..., m\}$, and

for a fixed value of m, the determinants in (1) alternate in sign as the index k runs from 1 to m. (2)

Establishment of these conjectures as theorems [2, 4, 5] and generalization of the theorems in turn to several other types of range spaces for the projection [7, 8] lead to the natural conjecture [6] that the Bernstein and Erdös conditions characterize interpolation of minimal norm into any finite-dimensional space spanned by an extended Tchebycheff system. While the answer to this general conjecture is as yet unknown, the following result provides a lower bound on the minimal norm of interpolation into any space spanned by an extended Tchebycheff system which satisfies the Bernstein and Erdös conditions by way of the determinant conditions (1) and (2).

THEOREM. *Let* $\mathbf{Y}$ *be an* $n+1$*-dimensional space spanned by an extended Tchebycheff system, interpolation into which satisfies conditions* (1) *and* (2). *Interpolation of minimal norm into* $\mathbf{Y}$ *has a norm exceeding that of optimal interpolation into the space of polynomials of degree* $n-1$ *or less.*

Proof. We note first of all that the formula

$$\partial X_i(T_i)/\partial t_j = -y_j(T_i)\, X_i'(t_j)$$

enables us to write the entries of the determinants in (1) and (2) explicitly.

At this point, a fairly standard sequence of cancellations can be carried out. It is possible to strike from the jth row of the determinants the denominator of y_j, for $j \in \{1,..., n-1\}$, and then, for $i \in \{1,..., n\}$, the ith column may be divided by the polynomial in T_i

$$(T_i - t_0) \cdots (T_i - t_n),$$

after which the ijth entry, which previously was

$$\partial X_i(T_i)/\partial t_j,$$

has now become

$$\frac{y_j(T_i)}{\prod_{k=0,k\neq j}^{n}(T_i - t_k)} \cdot \frac{X_i'(t_j)}{(t_j - T_i)}.$$

Now, conditions (1) and (2) imply that interpolation of minimal norm obeys Bernstein and Erdös and is characterized by the condition that

$$X_1(T_1) = \cdots = X_n(T_n),$$

and the nodes producing this are unique. We assume that we are at the optimal nodes. Conditions (1) and (2) also imply that, if the node t_{n-1} is moved toward t_0, the intervening nodes

$$t_1,..., t_{n-2}$$

may be varied implicitly, in such a manner that

$$X_1(T_1),..., X_{n-2}(T_{n-2})$$

retain their initial common value. Arguments provided by [3] demonstrate that the implicit function thus defined is global for

$$t_0 < t_1 < \cdots < t_n.$$

If we move t_{n-1} in this manner, the value $X_{n-1}(T_{n-1})$ must strictly decrease, and the value $X_n(T_n)$ at the same time must strictly increase. To see more clearly what is happening, we resort to an affine translation. We let $c \in [b, \infty)$ and perform interpolation on the interval $[a, c]$, letting $t_n = c$ increase without bound and holding t_{n-1} fixed. The nodes

$$t_1,..., t_{n-2}$$

may now be varied in the fixed interval $[t_0, t_{n-1}]$ as implicit functions of c. To carry out this procedure, we let $\tilde{\mathbf{Y}}(c)$ be the space of functions which are defined on $[a, c]$ by

$$\tilde{y}(s) = y(t) \qquad \text{whenever} \quad y \in \mathbf{Y},$$

and whenever

$$s = \left(\frac{t-a}{b-a}\right) b - a.$$

We define for $i \in \{0,..., n\}$

$\tilde{y}_i$ to be the ith fundamental function in $\tilde{\mathbf{Y}}$,

and for $i \in \{0,..., n-1\}$ we define the ith fundamental polynomial on the interval $[t_0, t_{n-1}]$ by

$$l_i(t) = \prod_{\substack{j=0 \\ j \neq i}}^{n-1} \frac{(t-t_j)}{(t_i - t_j)}.$$

Polynomial interpolation on the interval $[t_0, t_{n-1}]$ is defined in the obvious manner, using the fundamental functions

$$l_0,..., l_{n-1},$$

and its norm is determined in the manner described above for the more general space $\mathbf{Y}$.

We now let $t_n = c \to \infty$, holding the first $n-2$ maximum values

$$X_1(T_1),..., X_{n-2}(T_{n-2})$$

fixed as previously described. The value $X_{n-1}(T_{n-1})$ strictly decreases and is bounded below by 1. The proof is now completed by noting that, as $t_n = c \to \infty$, the following things occur:

(i) $\tilde{y}_i \to l_i$ uniformly and smoothly on $[t_0, t_{n-1}]$, for $i \in \{0,..., n-1\}$;

(ii) $\tilde{y}_n \to 0$ uniformly and smoothly on $[t_0, t_{n-1}]$.

Concluding Remarks. The above result raises some unanswered questions:

(i) Is the lower bound established here the best possible?

(ii) If so, is it attained?

(iii) What might be an *upper* bound on the minimal norm of interpolation for an extended Tchebycheff system?

Two simple examples may serve to illustrate the complexity of the situation. The first example is the space on the interval $[-1, 1]$, spanned by

$$\{e^{-t}, 1, e^t\}.$$

Optimal interpolation occurs on the nodes

$$t_0 = -1; \qquad t_1 = 0; \qquad t_2 = 1,$$

and the norm of optimal interpolation is less than 5/4, which is the norm of optimal interpolation with polynomials of degree 2 or less. More generally, we may interpolate with the span of

$$\{e^{-at}, 1, e^{at}\}, \qquad \text{for} \quad a > 0$$

on the same interval and same nodes. Because of the symmetry of this space, the nodes for interpolation of minimal norm are obviously -1, 0, and 1. The functions X_1 and X_2 are given by

$$X_1 = y_0 + y_1 - y_2 = 1 - 2y_2,$$

and

$$X_2 = -y_0 + y_1 + y_2 = 1 - 2y_0.$$

Since the maximum values of X_1 and X_2 on $[-1, 1]$ are equal, we note that

$$y_0(t) = (e^a + e^{a(t-1)} + e^{-at} - e^{at} - e^{-a} - e^{a(1-t)})/(2e^a - 2e^{-a} + e^{-2a} - e^{-2a}),$$

and

$$T_2 = 1/2, \qquad \text{regardless of the value of } a.$$

Thus,

$$y_0(T_2) = (e^a - e^{-a} + 2e^{-a/2} - 2e^{a/2})/(2e^a - 2e^{-a} + e^{-2a} - e^{2a}).$$

This expression approaches 0 as $a \to \infty$ and $-1/8$ as $a \to 0$, indicating that the norm of minimal interpolation fluctuates between the respective values of 1 and 5/4.

The second example is the space spanned by

$$\{1, t, \cos t\},$$

supported on $[-\pi/2, \pi/2]$. This space also has interpolation of minimal norm occurring when the nodes are located at the two ends and at the midpoint of the interval of interpolation, whenever the interval is of the form

$$[-a, a], \qquad \text{for} \quad 0 < a < \pi/2.$$

What is interesting in this case is that the norm of interpolation *exceeds* 5/4 and increases with a, decreasing to a lower limit of 5/4 as $a \to 0$.

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